# PRIMARY INTERSECTIONS FOR TWO SIDED IDEALS OF A NOETHERIAN MATRIX RING

BY
EDMUND H. FELLER

**Introduction.** The purpose of this paper is to obtain certain primary intersections as described in [1] for all two sided ideals in the matrix ring $D_n$ where $D$ is a Noetherian ring. We refer to such a ring as a Noetherian matrix ring. The primary intersections will depend only upon the Noetherian ring $D$. The following discussion will show that if the primary intersections of the ideals in $D$ are known one can immediately write primary intersections for all two sided ideals in $D_n$.

1. **Reformulation of theorems.** The following two theorems are reformulations of the author's Theorems 2.5 and 2.7 of [1]. The proofs are very similar to those of Noether [5] and Krull [4] and therefore were not included in [1] and are not included here. (See [2, pp. 172–181]). The definitions of [1] are used here.

THEOREM 1.1. *Let $N=N_1\cap \cdots \cap N_s=N_1^{\#}\cap \cdots \cap N_s^{\#}$ be irredundant intersections where $N_i$, $N_i^{\#}$ are irreducible $R$ submodules, $i=1, 2, \cdots, s$. Let $H$ be a subring of $\bigcap_{i=1}^{s} [V^*(N_i)\cap V^*(N_i^{\#})]$ containing the identity element, then the set of distinct $H$ radicals of $N_1, \cdots, N_s$ is identical with the set of distinct $H$ radicals of $N_1^{\#}, \cdots, N_s^{\#}$ in $H$.*

From this theorem and Theorem 2.6 of [1] we have

THEOREM 1.2. *Let $N$ be an $R$ submodule of the $A$-$R$ module $M$ which satisfies the A.C.C. for $R$ submodules*(1)*. Let $\alpha$ be an index that ranges over a possibly infinite set $G$ whose cardinal number is $\psi$ and let $N=\bigcap_{i=1}^{t} N_{i\alpha}$ be a set of $\psi$ irredundant representations of $N$ as the intersection of irreducible $R$ submodules $N_{i\alpha}$ of $M$. Let $H(G)$ be a subring with identity of $\bigcap V^*(N_{i\alpha})$ where $i$ ranges from 1 to $t$ and $\alpha$ ranges over $G$. Then for the $\alpha$th intersection there exist $H$ primary $R$ submodules $N'_{1\alpha}, \cdots, N'_{s\alpha}$ with distinct $H$ radicals $p_1, \cdots, p_s$ such that $N=N'_{1\alpha}\cap \cdots \cap N'_{s\alpha}$. If $N=N'_{1\beta}\cap \cdots \cap N'_{r\beta}$ is another such intersection where $\beta$ is an index of $G$ then $r=s$ and for a suitable rearrangement of the subscripts the corresponding $H$ radicals are equal.*

2. **Primary intersections for two sided ideals in $D_n$.** A ring with identity as a A-R module if one takes as $A$ the ring of left multiplications and as $R$ the ring of right multiplications. Thus theorems of this paper and [1] apply to rings with identity that satisfy the A.C.C. for right ideals.

Received by the editors April 22, 1957.

(1) An A-R module is defined as a right A, right R module in [3, page 17].

Let $D$ be a Noetherian ring. We shall consider the application of these theorems to all two sided ideals of $D_n$, the ring of $n$ by $n$ matrices with elements in $D$.

If $I$ is an ideal of $D$ then the set of all matrices $(a_{ij})$ with $a_{ij} \in D$ for $i \neq k$ and $a_{kj} \in I$ is a right ideal of $D_n$ which we shall denote by $(I, k)$.

STATEMENT 2.1. If $I$ is an ideal of $D$ and $(I, k)$ is contained in a right ideal $H$ of $D_n$ then $H$ is of the form $(I', k)$ where $I'$ is an ideal of $D$ which contains $I$.

**Proof.** Suppose as in the statement that $(I, k) \subseteq H$. The set of elements $I'$ that appear in the $k$th row of $H$ is an ideal of $D$. For suppose $a, b \in I'$ and say $a$ appears in the matrix $A$ of $H$ in the $(k, i)$ position and $b$ appears in the matrix $B$ of $H$ in the $(k, j)$ position. Let $E_{ij}$ denote the matrix with 1 in the $(i, j)$ position and zero elsewhere. Then $AE_{i1}+BE_{j1}$ is a matrix which contains $a+b$ in the $k$th row. If $c \in D$ then $AE_{i1}c$ is a matrix which contains $ac$ in the $k$th row. Next we shall show that if $c$ is an element of $I'$ then $H$ contains a matrix with $c$ in position $(k, 1)$ and zero elsewhere. Since $c \in I'$ there exist a matrix with $c$ in the $k$th row and by proper multiplication by the elements of $D_n$ on the right $H$ must contain a matrix $(a_{ij})$, $a_{i1} \in D$, $a_{k1}=c$, $a_{ij}=0$ for $j>1$. Since $(I, k) \subseteq H$, $H$ contains a matrix $(b_{ij})$, $b_{i1}=a_{i1}$, $b_{k1}=0$, $b_{ij}=0$ for $j>1$. Hence $(a_{ij})-(b_{ij})=(c_{ij})$ where $c_{1k}=c$, $c_{ij}=0$ for $i \neq 1$ and $j \neq k$. Consequently, since the right ideal $(0, k) \subseteq (I, k) \subseteq H$, we have $(c_{ij})+(0, k) \subseteq H$, i.e., the element $c$ appears in the $k$th row of the first column and hence in every column with all combinations of the elements of $D$ in the $j \neq k$ rows. Since this is true for all elements $c \in I'$, we have $H=(I', k)$.

STATEMENT 2.2. If $I$ is an irreducible(2) ideal of $D$, then $(I,k)$, $k=1, 2, \cdots, n$, is an irreducible(2) right ideal of $D_n$.

**Proof.** We shall prove that $(I, k)$ is irreducible. Suppose $(I, k)=H_1 \cap H_2$ where $H_1$ and $H_2$ are right ideals of $D_n$ which properly include $(I, k)$. Then by Statement 2.1, $H_1$ and $H_2$ are of the form $(I_1, k)$ and $(I_2, k)$, where $I_1$ and $I_2$ are ideals of $D$. Hence $I=I_1 \cap I_2$ where $I_1$ and $I_2$ properly include $I$—contradiction.

For an irreducible ideal $I$ of $D$ we have from [1, Theorem 2.1] and the preceding statement that $(I, k)$ is $V^*[(I, k)]$ primary, where $V^*[(I, k)]$ is the set of elements $A$ in $D_n$ such that $A(I, k) \subseteq (I, k)$, in the sense that if $AB \in (I, k)$, $B \notin (I, k)$, $A \in V^*[(I, k)]$, then $A^t \in (I, k)$ for some positive integer $t$.

Let $(I_1, I_2, k)$ denote the set of all matrices $(a_{ij})$ where $a_{ij} \in D$, $i \neq k$, $a_{kj} \in I_1$, $j \neq k$, and $a_{kk} \in I_2$ where $I_1$ and $I_2$ are ideals of $D$.

STATEMENT 2.3. $V^*[(I, k)]=(I, D, k)$.

**Proof.** Certainly $(I, D, k) \subseteq V^*[(I, k)]$. Suppose $A=(a_{ij}) \notin (I, D, k)$, say $a_{kj} \notin I$ for $j \neq k$. Let $E_{jk}$ denote the matrix with 1 in the $(j, k)$ position and zero

(2) An ideal here is irreducible in the sense that it is not the intersection of two right ideals which properly contain it.

elsewhere. Then for $E_{jk}$ contained in $(I, k)$ we have $AE_{jk} \notin (I, k)$ since $AE_{jk}$ contains $a_{kj}$ in the $(k, k)$ position. This proves the statement.

The $V^*$ radical of $(I, k)$ in $(I, D, k)$ is the set of matrices $A$ in $(I, D, k)$ such that $A^t \in (I, k)$ for some positive integer $t$.

STATEMENT 2.4. The $V^*$ radical of $(I, k)$ is $(I, P, k)$ where $P$ is the radical of the ideal $I$ in $D$. $(I, P, k)$ is a completely prime two sided ideal of $V^*$.

**Proof.** If $(a_{ij}) \in (I, D, k)$, then $(a_{ij})^t = (b_{ij})$ where $b_{ij} \in D$, $i \neq k$, $b_{kj} \in I$, $j \neq k$, $b_{kk} = c + a_{kk}^t$ for $c \in I$. Hence $(a_{ij})^t$ is contained in $(I, k)$ if and only if $a_{kk}^t \in I$ for some positive integer $t$, i.e., $a_{kk}$ is contained in the radical of $I$ in $D$. The second part follows from Theorem 2.2 of [1].

From [3, p. 40] the ideals of $D_n$ are of the form $I_n$ where $I$ is an ideal of $D$. From the irreducible intersections for $I$ in $D$ we can write irreducible intersections for $I_n$ in $D_n$. This is displayed in the next theorem the proof of which is most direct and is therefore omitted.

THEOREM 2.1. *Let $I_n$ be a two sided ideal in $D_n$ where $D$ is a Noetherian ring. Let $\alpha$ be an index that ranges over a possibly infinite set $E$ whose cardinal number is $\sigma$ and let $I = I_{1\alpha} \cap I_{2\alpha} \cap \cdots \cap I_{s\alpha}$ be a set of $\sigma$ irredundant representations of $I$ as an intersection of irreducible ideals $I_{i\alpha}$ in $D$. Then the equation*

$$I_n = \bigcap_{i=1}^{s} \bigcap_{k=1}^{n} (I_{i\alpha_k}, k) \tag{Z}$$

*in which, for each value of $k$, $\alpha_k$ is an arbitrary index from the set $E$, defines $\sigma^n$ representations of $I_n$ as an irredundant intersection of irreducible right ideals of $D_n$.*

For an intersection of the form (Z), since the $V^*$ radical of $(I_{i\alpha_k}, k)$ is $(I_{i\alpha_k}, P, k)$ where $P$ is the radical of $I_{i\alpha_k}$, these radicals will all be different. In addition if different ideals of $I$ are used in two intersections for $I_n$ of the form (Z) none of the $V^*$ radicals will be equal.

Let us now apply Theorems 2.6 of [1], 1.1, and 1.2 of this paper. For the $\sigma$ intersections of Theorem 2.1, consider as in Theorem 2.1 the set $S$ of $\sigma^n$ intersections for $I_n$ which can be formed from this set. Then $H = \bigcap V^*(I_{i\alpha}, k)$ where this intersection is taken over all $i$, $k$, and $\alpha$, which of course could be an infinite intersection. However for any one intersection of the form (Z) the intersection of the $n$ $V^*(I_{i\alpha}, k)$'s involved in this intersection is the set of all matrices $(a_{ij})$ where $a_{ij} \in I$ for $i \neq j$, $a_{ii} \in D$ which we shall denote by $(I \backslash D)$. Since this is true for all intersections of the form (Z) we have $H = (I \backslash D)$ which is a finite intersection. In general if $I_1$ and $I_2$ are ideals of $D$ we shall denote by $(I_1 \backslash I_2)$ the set of all matrices $(a_{ij})$ with $a_{ij} \in I_1$ for $i \neq j$, $a_{ii} \in I_2$. Then the radical of $(I_{i\alpha}, k)$ in $(I \backslash D)$ will be the set of all matrices $(a_{ij})$ where $a_{ij} \in I$, $i \neq j$, $a_{ii} \in D$ for $i \neq k$, $a_{kk} \in P$ where $P$ is the radical of $I_{i\alpha}$ in $D$. We shall denote such an ideal by $(I\backslash D, P, k)$(3). Consequently in $(I\backslash D)$ two radicals $(I\backslash D, P_1, k_1)$ and $(I\backslash D, P_2, k_2)$ will be equal if and only if $P_1 = P_2$ and $k_1 = k_2$. Thus we can apply Theorem 2.6 of [1] and combine the right ideals of (Z) which have the same radicals in $(I\backslash D)$. This will result in an intersection of primary $(I\backslash D)$ right ideals of $D_n$ which is equal to $I_n$.

Since two radicals $(I\backslash D, P_1, k)$ and $(I\backslash D, P_2, m)$ will be equal if and only if $P_1 = P_2$ and $k = m$ then in (Z) we have $(I_{i\alpha_k}, k) \cap (I_{j\alpha_m}, m)$ will be $(I\backslash D)$ primary if and only if $k = m$ and $I_{i\alpha_k} \cap I_{j\alpha_m}$ is a primary ideal of $D$. Thus in applying Theorem 2.6 of [1] we combine the right ideals of (Z) to write $(I\backslash D)$ primary intersections for $I_n$ with distinct $(I\backslash D)$ radicals.

From the previous discussion and Theorem 1.2 we have

THEOREM 2.2. *Let $I_n$ be a two sided ideal of $D_n$ where $D$ is a Noetherian ring. Let $\alpha$ be an index that ranges over a possibly infinite set $F$ whose cardinal number if $\rho$ and let $I = J_{1\alpha} \cap \cdots \cap J_{r\alpha}$ be a set of $\rho$ intersections for $I$ where $J_{i\alpha}$ are primary ideals of $D$. Then the equation*

$$I_n = \bigcap_{i=1}^{r} \bigcap_{k=1}^{n} (J_{i\alpha_k}, k) \tag{Y}$$

*in which, for each value of $k$, $\alpha_k$ is an arbitrary index from the set $F$, defines $\rho^n$ representations of $I_n$ as the intersection of $(I\backslash D)$ primary right ideals where the $(I\backslash D)$ radicals are distinct for each representation. For any two of these representations the $(I\backslash D)$ radicals are for some ordering equal.*

One may now ask: (1) Are the primary intersections for $I_n$ discussed here the only such intersections? (2) How can one write primary intersections for all right ideals of $D_n$?

## BIBLIOGRAPHY

1. E. H. Feller, *The lattice of submodules of a module over a noncommutative ring*, Trans. Amer. Math. Soc. vol. 81 (1956) pp. 342–357.

2. N. Jacobson, *Lectures in abstract algebra*, Van Nostrand Co., 1951.

3. ———, *Structure of rings*, Amer. Math. Soc. Colloquium Publications, vol. 37, 1956.

4. W. Krull, *Ein neuer Beweis für die Haupsätze der allgemeinen Idealtheorie*, Math. Ann. vol. 90 (1923) p. 55.

5. E. Noether, *Idealtheorie in Ringbereichen*, Math. Ann. vol. 83 (1921) pp. 24–66.

UNIVERSITY OF WISCONSIN-MILWAUKEE,
MILWAUKEE, WIS.

(3) The radical of $I$ in $(I\backslash D)$ will be equal to $(I\backslash P)$ where $P$ is the radical of $I$ in $D$.